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INVARIANCE OF THE WHITE NOISE FOR KDV

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Abstract. We prove the invariance of the mean 0 white noise for the periodic KdV. First, we show that the Besov-type space $\widehat{b}^s_{p,\infty}$, $sp < -1$, contains the support of the white noise. Then, we prove local well-posedness in $\widehat{b}^s_{p,\infty}$ for $p = 2+$, $s = -\frac{1}{2}+$ such that $sp < -1$. In establishing the local well-posedness, we use a variant of the Bourgain spaces with a weight. This provides an analytical proof of the invariance of the white noise under the flow of KdV obtained in Quastel-Valko [21].

1. Introduction

In this paper, we consider the periodic Korteweg-de Vries (KdV) equation:

$$\begin{cases} u_t + u_{xxx} + uu_x = 0 \\ u\big|_{t=0} = u_0, \end{cases} \tag{1}$$

where u is a real-valued function on $\mathbb{T} \times \mathbb{R}$ with $\mathbb{T} = [0, 2\pi)$ and the mean of u_0 is 0. By the conservation of the mean, it follows that the solution $u(t)$ of (1) has the spatial mean 0 for all $t \in \mathbb{R}$ as long as it exists. Our main goal is to show that the mean 0 white noise

$$d\mu = Z^{-1} \exp(-\tfrac{1}{2} \int u^2 dx) \prod_{x \in \mathbb{T}} du(x), \ \ u \text{ mean } 0 \tag{2}$$

is invariant under the flow and that (1) is globally well-posed almost surely on the statistical ensemble (i.e. on the support of μ) without using the complete integrability of the equation.

First, we briefly review recent well-posedness results of the periodic KdV (1). In [2], Bourgain introduced a new weighted space-time Sobolev space $X^{s,b}$ whose norm is given by

$$\|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} = \big\| \langle n \rangle^s \langle \tau - n^3 \rangle^b \widehat{u}(n, \tau) \big\|_{L^2_{n,\tau}(\mathbb{Z}\times\mathbb{R})}, \tag{3}$$

where $\langle \cdot \rangle = 1 + |\cdot|$. He proved the local well-posedness of (1) in $L^2(\mathbb{T})$ via the fixed point argument, immediately yielding the global well-posedness in $L^2(\mathbb{T})$ thanks to the conservation of the L^2 norm. Kenig-Ponce-Vega [14] improved Bourgain's result and established the local well-posedness in $H^{-\frac{1}{2}}(\mathbb{T})$. Colliander-Keel-Staffilani-Takaoka-Tao [9] proved the corresponding global well-posedness result via the I-method. More recently, Kappeler-Topalov [13] proved the global well-posedness of the KdV in $H^{-1}(\mathbb{T})$, using the complete integrability of the equation.

There are also results on the necessary conditions on the regularity with respect to smoothness or uniform continuity of the solution map : $u_0 \in H^s(\mathbb{T}) \to u(t) \in H^s(\mathbb{T})$. Bourgain [3] showed that if the solution map is C^3, then $s \geq -\frac{1}{2}$. Christ-Colliander-Tao

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[8] proved that if the solution map is uniformly continuous, then $s \geq -\frac{1}{2}$. (Also, see Kenig-Ponce-Vega [15].)

In [4], Bourgain proved the invariance of the Gibbs measures for the nonlinear Schrödinger equations (NLS). In dealing with the super-cubic nonlinearity, (where only the local well-posedness result was available), he used a probabilistic argument and the approximating finite dimensional ODEs (with the invariant finite dimensional Gibbs measures) to extend the local solutions to the global ones almost surely on the statistical ensemble and showed the invariance of the Gibbs measures. Note that it was crucial that the local well-posedness was obtained with a "good" estimate on the solutions (e.g. via the fixed point argument) for his argument to obtain the uniform convergence of the solutions of the finite dimensional ODEs to those of the full PDE. Also see Burq-Tzvetkov [6], Oh [19], and Tzvetkov [23], [24].

In the present paper, we'd like to follow Bourgain's argument [4]. Unfortunately, it is known (c.f. Zhidkov [25]) that the white noise μ in (2) is supported on $\cap_{s<-\frac{1}{2}} H^s \setminus H^{-\frac{1}{2}}$. In view of the results in [3] and [8] described above, we can not hope to have a local-in-time solution via the fixed point argument in H^s, $s < -\frac{1}{2}$. Instead, we will prove a local well-posedness in an appropriate Banach space containing the support of the white noise μ. Define the Besov-type space via the norm

$$\|f\|_{\widehat{b}^s_{p,\infty}} := \|\widehat{f}\|_{b^s_{p,\infty}} = \sup_j \|\langle n\rangle^s \widehat{f}(n)\|_{L^p_{|n|\sim 2^j}} = \sup_j \Big(\sum_{|n|\sim 2^j} \langle n\rangle^{sp} |\widehat{f}(n)|^p\Big)^{\frac{1}{p}}. \tag{4}$$

By Hausdorff-Young's inequality, we have $\widehat{b}^s_{p,\infty} \supset B^s_{p',\infty}$ for $p>2$, where $B^s_{p',\infty}$ is the usual Besov space with $p' = \frac{p}{p-1}$. In Section 3, we use the theory of abstract Wiener spaces to show that $\widehat{b}^s_{p,\infty}$ contains the full support of the white noise for $sp < -1$.

Now, we'd like to establish the local well-posedness in $\widehat{b}^s_{p,\infty}$ for $sp<-1$. Note that this space is essentially less regular than $H^{-\frac{1}{2}}$ since it contains the support of the white noise. First, define a variant of the $X^{s,b}$ space adjusted to $\widehat{b}^s_{p,\infty}(\mathbb{T})$. Let $X^{s,b}_p$ be the completion of the Schwartz class $\mathcal{S}(\mathbb{T}\times\mathbb{R})$ under the norm

$$\|u\|_{X^{s,b}_p} = \|\langle n\rangle^s \langle \tau - n^3\rangle^b \widehat{u}(n,\tau)\|_{b^s_{p,\infty} L^p_\tau}. \tag{5}$$

Then, one of the crucial bilinear estimates that we need to prove is:

$$\|\partial_x(uv)\|_{X^{s,-\frac{1}{2}}_p} \lesssim \|u\|_{X^{s,\frac{1}{2}}_p} \|v\|_{X^{s,\frac{1}{2}}_p}. \tag{6}$$

As in [2] and [14], a key ingredient is the algebraic identity $n^3 - n_1^3 - n_2^3 = 3nn_1n_2$ for $n = n_1 + n_2$. However, this is not enough to prove (6) for $sp<-1$. In establishing the local well-posedness through the usual integral equation, we view the nonlinear problem (1) as a perturbation to the Airy equation $u_t + u_{xxx} = 0$. Noting the Fourier transform of the solution to the Airy equation is a measure supported on $\{\tau = n^3\}$, we modify $X^{s,b}_p$ with a carefully chosen weight $w(n,\tau)$ in Section 4 to treat the resonant cases in (6). (c.f. Bejenaru-Tao [1], Kishimoto [17] in the context of NLS.)

Theorem 1. *Assume the mean 0 condition on u_0. Let $s = -\frac{1}{2}+$, $p = 2+$ such that $sp<-1$. Then, KdV (1) is locally well-posed in $\widehat{b}^s_{p,\infty}$.*

Once we prove Theorem 1, we can use the finite dimensional approximation to (1):

$$\begin{cases} u_t^N + u_{xxx}^N + \mathbb{P}_N(u^N u_x^N) = 0 \\ u^N\big|_{t=0} = u_0^N, \end{cases} \tag{7}$$

where $\mathbb{P}_\mathbb{N}$ is the projection onto the frequencies $|n| \leq N$ and $u^N = \mathbb{P}_N u$. Note that (7) is Hamiltonian, and that it preserves $\int (u^N)^2 dx$. Hence, by Liouville's theorem, the finite dimensional white noise

$$d\mu_N = Z_N^{-1} \exp(-\tfrac{1}{2} \int (u^N)^2 dx) \prod_{x \in \mathbb{T}} du^N(x) \tag{8}$$

is invariant under the flow of (7). The remaining argument follows just as in [4], [6], [19], [23], [24], and we obtain the a.s. GWP of (1) and the invariance of the white noise μ.

Theorem 2. *Let $\{g_n(\omega)\}_{n=1}^\infty$ be a sequence of i.i.d. standard complex Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$. Consider (1) with initial data $u_0 = \sum_{n\neq 0} g_n(\omega) e^{inx}$, where $g_{-n} = \overline{g_n}$. Then, (1) is globally well-posed almost surely in $\omega \in \Omega$. Moreover, the mean 0 white noise μ is invariant under the flow.*

Remark 1.1. This provides an analytical proof of the invariance of the white noise μ. Recently, Quastel-Valko [21] proved the invariance of the white noise under the flow of KdV. Their argument combines the GWP in $H^{-1}(\mathbb{T})$ via the complete integrability (Kappeler-Topalov [13]), the correspondence between the white noise for KdV and the Gibbs measure (weighted Wiener measure) of mKdV under the (corrected) Miura transform (Cambronero-McKean [7]), and the invariance of the Gibbs measure of mKdV (Bourgain [4].) Their method is not applicable to the general non-integrable coupled KdV system considered in [19], whereas our argument is applicable in the non-integrable case as well.

Remark 1.2. *Let $\mathcal{F}L^{s,p}$ be the space of functions on $\mathbb{T}$ defined via the norm $\|f\|_{\mathcal{F}L^{s,p}} = \|\langle n\rangle^s \widehat{f}(n)\|_{L^p_n}$. Then, Theorems 1 and 2 can also be established in $\mathcal{F}L^{s,p}$ for some $s = -\frac{1}{2}+$, $p = 2+$ with $sp < -1$. See Remark 4.7.*

This paper is organized as follows: In Section 2, we introduce some standard notations. In Section 3, we go over the basic theory of Gaussian Hilbert spaces and abstract Wiener spaces. Then, we give the precise mathematical meaning to the white noise μ and show that it is a (countably additive) probability measure on $\widehat{b}^s_{p,\infty}$ for $sp < -1$. In Section 4, we introduce the function spaces and linear estimates. Then, we prove Theorem 1 by establishing the crucial bilinear estimate.

2. Notation

In the periodic setting on $\mathbb{T}$, the spatial Fourier domain is $\mathbb{Z}$. Let dn be the normalized counting measure on $\mathbb{Z}$, and we say $f \in L^p(\mathbb{Z})$, $1 \leq p < \infty$, if

$$\|f\|_{L^p(\mathbb{Z})} = \bigg(\int_\mathbb{Z} |f(n)|^p dn\bigg)^{\frac{1}{p}} := \bigg(\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} |f(n)|^p\bigg)^{\frac{1}{p}} < \infty.$$

If $p = \infty$, we have the obvious definition involving the essential supremum. We often drop 2π for simplicity. If a function depends on both x and t, we use $^{\wedge_x}$ (and $^{\wedge_t}$) to denote the spatial (and temporal) Fourier transform, respectively. However, when there is no confusion, we simply use $^\wedge$ to denote the spatial Fourier transform, the temporal Fourier transform, and the space-time Fourier transform, depending on the context.

Given a space X of functions on $\mathbb{T}\times\mathbb{R}$, we define the local in time restriction $X(\mathbb{T}\times I)$ for any time interval $I=[t_1,t_2]\subset\mathbb{R}$, (or simply $X_{[t_1,t_2]}$) by

$$\|u\|_{X_I}=\|u\|_{X(\mathbb{T}\times I)}=\inf\big\{\|\widetilde{u}\|_{X(\mathbb{T}\times\mathbb{R})}:\widetilde{u}|_I=u\big\}.$$

For a Banach space $X\subset\mathcal{S}'(\mathbb{T}\times\mathbb{R})$, we use $\widehat{X}$ to denote the space of the Fourier transforms of the functions in X, which is a Banach space with the norm $\|f\|_{\widehat{X}}=\|\mathcal{F}^{-1}_{n,\tau}f\|_X$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform (in n and τ.) Also, for a space Y of functions on $\mathbb{Z}$, we use $\widehat{Y}$ to denote the space of the inverse Fourier transforms of the functions in Y with the norm $\|f\|_{\widehat{Y}}=\|\mathcal{F}f\|_Y$.

Now, define $\widehat{b}^s_{p,q}(\mathbb{T})$ by the norm

$$\|f\|_{\widehat{b}^s_{p,q}(\mathbb{T})}=\|\widehat{f}\|_{b^s_{p,q}(\mathbb{Z})}:=\big\|\|\langle n\rangle^s\widehat{f}(n)\|_{L^p_{|n|\sim 2^j}}\big\|_{l^q_j}=\Big(\sum_{j=0}^\infty\Big(\sum_{|n|\sim 2^j}\langle n\rangle^{sp}|\widehat{f}(n)|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} \tag{9}$$

for $q<\infty$ and by (4) when $q=\infty$.

Lastly, let $\eta\in C_c^\infty(\mathbb{R})$ be a smooth cutoff function supported on $[-1,1]$ with $\eta\equiv 1$ on $[-\frac12,\frac12]$ and let $\eta_{_T}(t)=\eta(T^{-1}t)$. We use c, C to denote various constants, usually depending only on s and p. If a constant depends on other quantities, we will make it explicit. We use $A\lesssim B$ to denote an estimate of the form $A\leq CB$. Similarly, we use $A\sim B$ to denote $A\lesssim B$ and $B\lesssim A$ and use $A\ll B$ when there is no general constant C such that $B\leq CA$. We also use $a+$ (and $a-$) to denote $a+\varepsilon$ (and $a-\varepsilon$), respectively, for arbitrarily small $\varepsilon\ll 1$.

3. Gaussian Measures in Hilbert Spaces and Abstract Wiener Spaces

In this section, we go over the basic theory of Gaussian measures in Hilbert spaces and abstract Wiener spaces to provide the precise meaning of the white noise "$d\mu=Z^{-1}\exp(-\frac12\int u^2dx)\prod_{x\in\mathbb{T}}du(x)$" appearing in Section 1. For details, see Zhidokov [25], Gross [12], and Kuo [18].

First, recall (centered) Gaussian measures in $\mathbb{R}^n$. Let $n\in\mathbb{N}$ and B be a symmetric positive $n\times n$ matrix with real entries. The Borel measure μ in $\mathbb{R}^n$ with the density

$$d\mu(x)=\frac{1}{\sqrt{(2\pi)^n\det(B)}}\exp\big(-\tfrac12\langle B^{-1}x,x\rangle_{\mathbb{R}^n}\big)$$

is called a (nondegenerate centered) Gaussian measure in $\mathbb{R}^n$. Note that $\mu(\mathbb{R}^n)=1$.

Now, we consider the analogous definition of the infinite dimensional (centered) Gaussian measures. Let H be a real separable Hilbert space and $B:H\to H$ be a linear positive self-adjoint operator (generally not bounded) with eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ and the corresponding eigenvectors $\{e_n\}_{n\in\mathbb{N}}$ forming an orthonormal basis of H. We call a set $M\subset H$ cylindrical if there exists an integer $n\geq 1$ and a Borel set $F\subset\mathbb{R}^n$ such that

$$M=\big\{x\in H:(\langle x,e_1\rangle_H,\cdots,\langle x,e_n\rangle_H)\in F\big\}. \tag{10}$$

For a fixed operator B as above, we denote by $\mathcal{A}$ the set of all cylindrical subsets of H. One can easily verify that $\mathcal{A}$ is a field. Then, the centered Gaussian measure in H with the correlation operator B is defined as the additive (but not countably additive in general) measure μ defined on the field $\mathcal{A}$ via

$$\mu(M)=(2\pi)^{-\frac{n}{2}}\prod_{j=1}^n\lambda_j^{-\frac12}\int_F e^{-\frac12\sum_{j=1}^n\lambda_j^{-1}x_j^2}dx_1\cdots dx_n,\ \text{for } M\in\mathcal{A}\text{ as in (10).} \tag{11}$$

The following theorem tells us when this Gaussian measure μ is countably additive.

Theorem 3.1. *The Gaussian measure μ defined in* (11) *is countably additive on the field $\mathcal{A}$ if and only if B is an operator of trace class, i.e. $\sum_{n=1}^{\infty} \lambda_n < \infty$. If the latter holds, then the minimal σ-field $\mathcal{M}$ containing the field $\mathcal{A}$ of all cylindrical sets is the Borel σ-field on H.*

Consider a sequence of the finite dimensional Gaussian measures $\{\mu_n\}_{n\in\mathbb{N}}$ as follows. For fixed $n \in \mathbb{N}$, let $\mathcal{M}_n$ be the set of all cylindrical sets in H of the form (10) with this fixed n and arbitrary Borel sets $F \subset \mathbb{R}^n$. Clearly, $\mathcal{M}_n$ is a σ-field, and setting

$$\mu_n(M) = (2\pi)^{-\frac{n}{2}} \prod_{j=1}^{n} \lambda_j^{-\frac{1}{2}} \int_F e^{-\frac{1}{2}\sum_{j=1}^{n} \lambda_j^{-1} x_j^2} dx_1 \cdots dx_n$$

for $M \in \mathcal{M}_n$, we obtain a countably additive measure μ_n defined on $\mathcal{M}_n$. Then, one can show that each measure μ_n can be naturally extended onto the whole Borel σ-field $\mathcal{M}$ of H by $\mu_n(A) := \mu_n(A \cap \operatorname{span}\{e_1, \cdots, e_n\})$ for $A \in \mathcal{M}$. Then, we have

Proposition 3.2. *Let μ in* (11) *be countably additive. Then, $\{\mu_n\}_{n\in\mathbb{N}}$ constructed above converges weakly to μ as $n \to \infty$.*

Now, we construct the mean 0 white noise. Let $\phi = \sum_n a_n e^{inx}$ be a real-valued function on $\mathbb{T}$ with mean 0. i.e. we have $a_0 = 0$ and $a_{-n} = \overline{a_n}$. First, define μ_N on $\mathbb{C}^N \cong \mathbb{R}^{2N}$ with the density

$$d\mu_N = Z_N^{-1} e^{-\frac{1}{2}\sum_{n=1}^{N} |a_n|^2} \textstyle\prod_{n=1}^{N} da_n, \tag{12}$$

where $Z_N = \int_{\mathbb{C}^N} e^{-\frac{1}{2}\sum_{n=1}^{N} |a_n|^2} \prod_{n=1}^{N} da_n$. Note that this measure is the induced probability measure on $\mathbb{C}^N$ under the map $\omega \mapsto \{g_n(\omega)\}_{n=1}^{N}$, where $g_n(\omega)$, $n = 1, \cdots, N$, are i.i.d. standard complex Gaussian random variables. Next, define the white noise μ by

$$d\mu = Z^{-1} e^{-\frac{1}{2}\sum_{n\geq 1} |a_n|^2} \textstyle\prod_{n\geq 1} da_n, \tag{13}$$

where $Z = \int e^{-\frac{1}{2}\sum_{n\geq 1} |a_n|^2} \prod_{n\geq 1} da_n$. Then, in the above correspondence, we have $\phi = \sum_{n\neq 0} g_n e^{inx}$, where $\{g_n(\omega)\}_{n\geq 1}$ are i.i.d. standard complex Gaussian random variables and $g_{-n} = \overline{g_n}$.

Let $\dot{H}_0^s$ be the homogeneous Sobolev space restricted to the *real-valued* mean 0 elements. Let $\langle \cdot, \cdot \rangle_{\dot{H}_0^s}$ denote the inner product in $\dot{H}_0^s$. i.e. $\big\langle \sum c_n e^{inx}, \sum d_n e^{inx} \big\rangle_{\dot{H}_0^s} = \sum_{n\geq 1} |n|^{2s} c_n \overline{d_n}$. Let $B_s = \sqrt{-\Delta}^{2s}$. Then, the weighted exponentials $\{|n|^{-s} e^{inx}\}_{n\neq 0}$ are the eigenvectors of B_s with the eigenvalue $|n|^{2s}$, forming an orthonormal basis of $\dot{H}_0^s$. Note that

$$-\tfrac{1}{2}\langle B^{-1}\phi, \phi \rangle_{\dot{H}_0^s} = -\tfrac{1}{2}\Big\langle \sum_{n\neq 0} |n|^{-2s} a_n e^{inx}, \sum_{n\neq 0} a_n e^{inx} \Big\rangle_{\dot{H}_0^s} = -\tfrac{1}{2}\sum_{n\geq 1} |a_n|^2.$$

The right hand side is exactly the expression appearing in the exponent in (13). By Theorem 3.1, μ is countably additive if and only if B is of trace class, i.e. $\sum_{n\neq 0} |n|^{2s} < \infty$. Hence, $\bigcap_{s<-\frac{1}{2}} H^s$ is a natural space to work on. Unfortunately, the results in [3] and [8] state that one can not have a local-in-time solution of (1) via the fixed point argument in H^s, $s < -\frac{1}{2}$. Instead, we propose to work on $\widehat{b}^s_{p,\infty}(\mathbb{T})$ defined in (4) for $sp < -1$ in view of Theorem 1. Since $\widehat{b}^s_{p,\infty}$ is not a Hilbert space, we need to go over the basic theory of abstract Wiener spaces.

Recall the following definitions [18]: Given a real separable Hilbert space H with norm $\|\cdot\|$, let $\mathcal{F}$ denote the set of finite dimensional orthogonal projections $\mathbb{P}$ of H. Then, define a cylinder set E by $E = \{x \in H : \mathbb{P}x \in F\}$ where $\mathbb{P} \in \mathcal{F}$ and F is a Borel subset of $\mathbb{P}H$, and let $\mathcal{R}$ denote the collection of such cylinder sets. Note that $\mathcal{R}$ is a field but not a σ-field. Then, the Gauss measure μ on H is defined by

$$\mu(E) = (2\pi)^{-\frac{n}{2}} \int_F e^{-\frac{\|x\|^2}{2}} dx$$

for $E \in \mathcal{R}$, where $n = \dim \mathbb{P}H$ and dx is the Lebesgue measure on $\mathbb{P}H$. It is known that μ is finitely additive but not countably additive in $\mathcal{R}$.

A seminorm $|||\cdot|||$ in H is called measurable if for every $\varepsilon > 0$, there exists $\mathbb{P}_0 \in \mathcal{F}$ such that

$$\mu(|||\mathbb{P}x||| > \varepsilon) < \varepsilon$$

for $\mathbb{P} \in \mathcal{F}$ orthogonal to $\mathbb{P}_0$. Any measurable seminorm is weaker than the norm of H, and H is not complete with respect to $|||\cdot|||$ unless H is finite dimensional. Let B be the completion of H with respect to $|||\cdot|||$ and denote by i the inclusion map of H into B. The triple (i, H, B) is called an abstract Wiener space.

Now, regarding $y \in B^*$ as an element of $H^* \equiv H$ by restriction, we embed B^* in H. Define the extension of μ onto B (which we still denote by μ) as follows. For a Borel set $F \subset \mathbb{R}^n$, set

$$\mu(\{x \in B : ((x, y_1), \cdots, (x, y_n)) \in F\}) := \mu(\{x \in H : (\langle x, y_1\rangle_H, \cdots, \langle x, y_n\rangle_H) \in F\}),$$

where y_j's are in B^* and $(\cdot, \cdot)$ denote the natural pairing between B and B^*. Let $\mathcal{R}_B$ denote the collection of cylinder sets $\{x \in B : ((x, y_1), \cdots, (x, y_n)) \in F\}$ in B.

Theorem 3.3 (Gross [12])**.** *μ is countably additive in the σ-field generated by $\mathcal{R}_B$.*

In the present context, let $H = L^2(\mathbb{T})$ and $B = \widehat{b}^s_{p,\infty}(\mathbb{T})$ for $sp < -1$. Then, we have

Proposition 3.4. *The seminorms $\|\cdot\|_{\widehat{b}^s_{p,\infty}}$ is measurable for $sp < -1$.*

Hence, $(i, H, B) = (i, L^2, \widehat{b}^s_{p,\infty})$ is an abstract Wiener space, and μ defined in (13) is countably additive in $\widehat{b}^s_{p,\infty}$. We present the proof of Proposition 3.4 at the end of this section. It seems that the statement in Proposition 3.4 holds true for $sp = -1$ (c.f. Roynette [22] for $p = 2$.) However, we can choose s and p such that $sp < -1$ for our application, and thus we will not discuss the endpoint case. It follows from the proof that $(i, L^2, \mathcal{F}L^{s,p})$, where $\mathcal{F}L^{s,p} = \widehat{b}^s_{p,p}$, is also an abstract Wiener space for $sp < -1$ (we need a strict inequality in this case.)

Given an abstract Wiener space (i, H, B), we have the following integrability result due to Fernique [10].

Theorem 3.5 (Theorem 3.1 in [18])**.** *Let (i, H, B) be an abstract Wiener space. Then, there exists $c > 0$ such that $\int_B e^{c\|x\|^2_B} \mu(dx) < \infty$. Hence, there exists $c' > 0$ such that $\mu(\|x\|_B > K) \leq e^{-c'K^2}$.*

In our context, if $sp < -1$, we have $\mu\big(\|\phi\|_{\widehat{b}^s_{p,\infty}(\mathbb{T})} \geq K, \phi \text{ mean } 0\big) \leq e^{-cK^2}$ for some $c > 0$. With this estimate and Theorem 1, we can follow the argument in [4] to prove Theorem 2. We omit the details. Also, see [6], [19], [23], [24] for the details.

Proof of Proposition 3.4. We present the proof only for $2 < p < \infty$, which is the relevant case for our application. We just point out that the proof for $p \leq 2$ is similar but simpler (where one can use Hölder inequality in place of Lemma 3.6 below.) For $p = \infty$, see [4], [5], [19].

It suffices to show that for given $\varepsilon > 0$, there exists large M_0 such that

$$\mu\big(\|\mathbb{P}_{>M_0}\phi\|_{\widehat{b}^s_{p,\infty}} > \varepsilon\big) < \varepsilon,$$

where $\mathbb{P}_{>M_0}$is the projection onto the frequencies $|n| > M_0$. In the following, write $\phi = \sum_{n\neq 0} g_n e^{inx}$, where $\{g_n(\omega)\}_{n=1}^\infty$ is a sequence of i.i.d. standard complex-valued Gaussian random variables and $g_{-n} = \overline{g_n}$. First, recall the following lemma.

Lemma 3.6 (Lemma 4.7 in [20]). *Let $\{g_n\}$ be a sequence of i.i.d standard complex-valued Gaussian random variables. Then, for M dyadic and $\delta > 0$, we have*

$$\lim_{M\to\infty} M^{1-\delta}\frac{\max_{|n|\sim M}|g_n|^2}{\sum_{|n|\sim M}|g_n|^2} = 0, \quad a.s.$$

Fix $K > 1$ and $\delta \in (0, \frac{1}{2})$ (to be chosen later.) Then, by Lemma 3.6 and Egoroff's theorem, there exists a set E such that $\mu(E^c) < \frac{1}{2}\varepsilon$ and the convergence in Lemma 3.6 is uniform on E. i.e. we can choose dyadic M_0 large enough such that

$$\frac{\|\{g_n(\omega)\}_{|n|\sim M}\|_{L^\infty_n}}{\|\{g_n(\omega)\}_{|n|\sim M}\|_{L^2_n}} \leq M^{-\delta}, \tag{14}$$

for all $\omega \in E$ and dyadic $M > M_0$. In the following, we will work only on E and drop '$\cap E$' for notational simplicity. However, it should be understood that all the events are under the intersection with E so that (14) holds.

Let $\{\sigma_j\}_{j\geq 1}$ be a sequence of positive numbers such that $\sum \sigma_j = 1$, and let $M_j = M_0 2^j$ dyadic. Note that $\sigma_j = C2^{-\lambda j} = CM_0^\lambda M_j^{-\lambda}$ for some small $\lambda > 0$ (to be determined later.) Then, we have

$$\begin{aligned}\mu\big(\|\mathbb{P}_{>M_0}\phi\|_{\widehat{b}^s_{p,\infty}} > \varepsilon\big) &\leq \mu\big(\|\{g_n\}_{|n|>M_0}\|_{b^s_{p,1}} > \varepsilon\big)\\ &\leq \sum_{j=0}^\infty \mu\big(\|\{\langle n\rangle^s g_n\}_{|n|\sim M_j}\|_{L^p_n} > \sigma_j\varepsilon\big),\end{aligned} \tag{15}$$

where $b^s_{p,1}$ is defined in (9). By interpolation and (14), we have

$$\begin{aligned}\|\{\langle n\rangle^s g_n\}_{|n|\sim M_j}\|_{L^p_n} &\sim M_j^s\|\{g_n\}_{|n|\sim M_j}\|_{L^p_n} \leq M_j^s\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n}^{\frac{2}{p}}\|\{g_n\}_{|n|\sim M_j}\|_{L^\infty_n}^{\frac{p-2}{p}}\\ &\leq M_j^s\|\{g_n\}_{|n|\sim M}\|_{L^2_n}\left(\frac{\|\{g_n\}_{|n|\sim M_j}\|_{L^\infty_n}}{\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n}}\right)^{\frac{p-2}{p}} \leq M_j^{s-\delta\frac{p-2}{p}}\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n}\end{aligned}$$

a. s. Thus, if we have $\|\{\langle n\rangle^s g_n\}_{|n|\sim M_j}\|_{L^p_n} > \sigma_j\varepsilon$, then we have $\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n} \gtrsim R_j$ where $R_j := \sigma_j\varepsilon M_j^{-s+\delta\frac{p-2}{p}}$. With $p = 2 + 2\theta$, we have $-s + \delta\frac{p-2}{p} = \frac{-sp+2\delta\theta}{2+2\theta} > \frac{1}{2}$ by taking δ sufficiently close to $\frac{1}{2}$ since $-sp > 1$. Then, by taking $\lambda > 0$ sufficiently small, $R_j = \sigma_j\varepsilon M_j^{-s+\delta\frac{p-2}{p}} = C\varepsilon M_0^\lambda M_j^{-s+\delta\frac{p-2}{p}-\lambda} \gtrsim C\varepsilon M_0^\lambda M_j^{\frac{1}{2}+}$. By a direct computation in the polar coordinates, we have

$$\mu\big(\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n} \gtrsim R_j\big) \sim \int_{B^c(0,R_j)} e^{-\frac{1}{2}|g|^2} \prod_{|n|\sim M_j} dg_n \lesssim \int_{R_j}^\infty e^{-\frac{1}{2}r^2} r^{2\cdot\#\{|n|\sim M_j\}-1} dr.$$

Note that the implicit constant in the inequality is $\sigma(S^{2\cdot\#\{|n|\sim M_j\}-1})$, a surface measure of the $2\cdot\#\{|n|\sim M_j\}-1$ dimensional unit sphere. We drop it since $\sigma(S^n) = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2}) \lesssim 1$. By change of variable $t = M_j^{-\frac{1}{2}} r$, we have $r^{2\cdot\#\{|n|\sim M_j\}-2} \lesssim r^{4M_j} \sim M_j^{2M_j} t^{4M_j}$. Since $t > M_j^{-\frac{1}{2}} R_j = C\varepsilon M_0^\lambda M_j^{0+}$, we have

$$M_j^{2M_j} = e^{2M_j \ln M_j} < e^{\frac{1}{8}M_j t^2} \quad \text{and} \quad t^{4M_j} < e^{\frac{1}{8}M_j t^2}$$

for M_0 sufficiently large. Thus, we have $r^{2\cdot\#\{|n|\sim M_j\}-2} < e^{\frac{1}{4}M_j t^2} = e^{\frac{1}{4}r^2}$ for $r > R$. Hence, we have

$$\mu\big(\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n} \gtrsim R_j\big) \le C \int_{R_j}^\infty e^{-\frac{1}{4}r^2} r\,dr \le e^{-cR_j^2} = e^{-cC^2 M_0^{2\lambda} M_j^{1+}\varepsilon^2}. \tag{16}$$

From (15) and (16), we have

$$\mu\big(\|\mathbb{P}_{>M_0}\phi\|_{\widehat{b}^s_{p,\infty}} > \varepsilon\big) \le \sum_{j=1}^\infty e^{-cC^2 M_0^{1+2\lambda+} 2^{j+}\varepsilon^2} \le \tfrac{1}{2}\varepsilon$$

by choosing M_0 sufficiently large. □

4. Local Well-Posedness in $\widehat{b}^s_{p,\infty}$

In this section, we prove Theorem 1 via the fixed point argument. In Subsection 4.1, we go over the previous local well-posedness theory of KdV to motivate the definition of the Bourgain space $W_p^{s,b}$ with the weight, adjusted to $\widehat{b}^s_{p,\infty}$. Then, we establish the basic linear estimates in Subsection 4.2. Finally, we prove the crucial bilinear estimate in Subsection 4.3.

4.1. **Bourgain Space with a weight.** In [14], Kenig-Ponce-Vega proved

$$\|\partial_x(uv)\|_{X^{s,-\frac{1}{2}}} \lesssim \|u\|_{X^{s,\frac{1}{2}}} \|v\|_{X^{s,\frac{1}{2}}}, \tag{17}$$

for $s \ge -\frac{1}{2}$ under the mean 0 assumption on u and v, where $X^{s,b}$ is defined in (3). Their proof is based on proving the equivalent statement:

$$\|B_s(f,g)\|_{L^2_{n,\tau}} \lesssim \|f\|_{L^2_{n,\tau}} \|g\|_{L^2_{n,\tau}} \tag{18}$$

where $B_s(\cdot,\cdot)$ is defined by

$$B_s(f,g)(n,\tau) = \frac{1}{2\pi\langle\tau-n^3\rangle^{\frac{1}{2}}} \sum_{\substack{n_1+n_2=n\\ n_1\neq 0,n}} \frac{|n|\langle n\rangle^s}{\langle n_1\rangle^s\langle n_2\rangle^s} \int_{\tau_1+\tau_2=\tau} \frac{f(n_1,\tau_1)g(n_2,\tau_2)d\tau_1}{\langle\tau_1-n_1^3\rangle^{\frac{1}{2}}\langle\tau_2-n_2^3\rangle^{\frac{1}{2}}}. \tag{19}$$

One of the main ingredients is the observation due to Bourgain [2]:

$$n^3 - n_1^3 - n_2^3 = 3nn_1n_2, \text{ for } n = n_1+n_2, \tag{20}$$

which in turn implies that

$$\text{MAX} := \max(\langle\tau-n^3\rangle, \langle\tau_1-n_1^3\rangle, \langle\tau_2-n_2^3\rangle) \gtrsim \langle nn_1n_2\rangle \tag{21}$$

for $n = n_1 + n_2$ and $\tau = \tau_1 + \tau_2$ with $n, n_1, n_2 \neq 0$. Recall that (21) implies that

$$\frac{|n|\langle n\rangle^s}{\langle n_1\rangle^s\langle n_2\rangle^s}\frac{1}{\langle \tau - n^3\rangle^{\frac{1}{2}}\langle \tau_1 - n_1^3\rangle^{\frac{1}{2}}\langle \tau_2 - n_2^3\rangle^{\frac{1}{2}}} \lesssim \frac{|n|\langle n\rangle^s}{\langle n_1\rangle^s\langle n_2\rangle^s}\frac{1}{\text{MAX}^{\frac{1}{2}}} \lesssim 1 \tag{22}$$

for $s \geq -\frac{1}{2}$. Note that (22) is optimal, for example, when $\langle \tau - n^3\rangle \sim \langle 3nn_1n_2\rangle$ and $\langle \tau_j - n_j^3\rangle \ll \langle 3nn_1n_2\rangle^{0+}$. To exploit this along with the fact the free solution concentrates on the curve $\{\tau = n^3\}$, we define the weight $w(n,\tau)$ in the following.

For $k \in \mathbb{Z}\setminus\{0\}$, let

$$A_k = \{(n,\tau) : |n| \geq C, \langle \tau - n^3 + 3n(n-k)k\rangle \ll \langle n\rangle^{\frac{1}{100}}\},$$

for some $C > 0$. With $\delta = 0+$ (to be determined later), let

$$w(n,\tau) = 1 + \sum_{k\neq 0}\min(\langle k\rangle, \langle n-k\rangle)^\delta \chi_{A_k}. \tag{23}$$

Note that, for fixed n and τ, there are at most two values of k such that $|(n-k)k + \frac{\tau - n^3}{3n}| \ll \langle n\rangle^{-1+\frac{1}{100}}$. It follows from the definition that $w(n,\tau) \lesssim \max(1, (\frac{\langle \tau - n^3\rangle}{\langle n\rangle})^{0+}) \leq \langle \tau - n^3\rangle^{0+}$.

Now, define the Bourgain space $W_p^{s,b}$ with the weight w via the norm

$$\|u\|_{W_p^{s,b}} = \|\widehat{u}\|_{\widehat{W}_p^{s,b}} := \|w\widehat{u}\|_{\widehat{X}_p^{s,b}} + \|\widehat{u}\|_{\widehat{Y}_p^{s,b-\frac{1}{2}}}, \tag{24}$$

where

$$\begin{cases} \|f\|_{\widehat{X}_p^{s,b}} := \|\langle n\rangle^s\langle \tau - n^3\rangle^b f(n,\tau)\|_{b^0_{p,\infty}L^p_\tau} = \sup_j \|\langle n\rangle^s\langle \tau - n^3\rangle^b f(n,\tau)\|_{L^p_{|n|\sim 2^j}L^p_\tau} \\ \|f\|_{\widehat{Y}_p^{s,b}} := \|\langle n\rangle^s\langle \tau - n^3\rangle^b f(n,\tau)\|_{b^0_{p,\infty}L^1_\tau} = \sup_j \|\langle n\rangle^s\langle \tau - n^3\rangle^b f(n,\tau)\|_{L^p_{|n|\sim 2^j}L^1_\tau}. \end{cases}$$

For our application, we set $b = \frac{1}{2}$. Note that $Y_p^{s,0}$ is introduced so that we have $W_p^{s,\frac{1}{2}}(\mathbb{T}\times[-T,T]) \subset C([-T,T]; \widehat{b}^s_{p,\infty}(\mathbb{T}))$. In the following, we take $p > 2$.

4.2. **Linear Estimates.** Let $S(t) = e^{-t\partial_x^3}$ and $\eta(t)$ be a smooth cutoff such that $\eta(t) = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $= 0$ for $|t| \geq 1$.

Lemma 4.1. *For any $s \in \mathbb{R}$, we have* $\|\eta(t)S(t)u_0\|_{W_p^{s,\frac{1}{2}}} \lesssim \|u_0\|_{\widehat{b}^s_{p,\infty}}$.

Proof. Recall that $w(n,\tau) \lesssim \langle \tau - n^3\rangle^{0+}$. Noting that $(\eta(t)S(t)u_0)^\wedge(n,\tau) = \widehat{\eta}(\tau - n^3)\widehat{u_0}(n)$, we have

$$\begin{aligned}\|\eta(t)S(t)u_0\|_{W_p^{s,\frac{1}{2}}} &\leq \sup_j \big\|\langle n\rangle^s\|\langle \tau - n^3\rangle^{\frac{1}{2}+}\widehat{\eta}(\tau - n^3)\|_{L^p_\tau}|\widehat{u_0}(n)|\big\|_{L^p_{|n|\sim 2^j}} \\ &\quad + \sup_j \big\|\langle n\rangle^s\|\widehat{\eta}(\tau - n^3)\|_{L^1_\tau}|\widehat{u_0}(n)|\big\|_{L^p_{|n|\sim 2^j}} \leq C_\eta\|u_0\|_{\widehat{b}^s_{p,\infty}},\end{aligned}$$

where $C_\eta = \|\langle \tau\rangle^{\frac{1}{2}+}\widehat{\eta}(\tau)\|_{L^p_\tau} + \|\widehat{\eta}\|_{L^1} < \infty$. □

Now, we estimate the Duhamel term. By the standard computation [2], we have

$$\begin{aligned} \int_0^t S(t-t')F(x,t')dt' = & -i\sum_{k=1}^\infty \frac{i^k t^k}{k!}\sum_{n\neq 0} e^{i(nx+n^3t)}\int \eta(\lambda-n^3)\widehat{F}(n,\lambda)d\lambda \\ & + i\sum_{n\neq 0} e^{inx}\int \frac{\big(1-\eta\big)(\lambda-n^3)}{\lambda-n^3}e^{i\lambda t}\widehat{F}(n,\lambda)d\lambda \\ & + i\sum_{n\neq 0} e^{i(nx+n^3t)}\int \frac{\big(1-\eta\big)(\lambda-n^3)}{\lambda-n^3}\widehat{F}(n,\lambda)d\lambda \\ =: & \, \mathcal{N}_1(F)(x,t)+\mathcal{N}_2(F)(x,t)+\mathcal{N}_3(F)(x,t). \end{aligned} \tag{25}$$

Lemma 4.2. *For any $s \in \mathbb{R}$, we have*

$$\|\eta(t)\mathcal{N}_1(F)\|_{W_p^{s,\frac{1}{2}}},\ \|\mathcal{N}_2(F)\|_{W_p^{s,\frac{1}{2}}}, \|\eta(t)\mathcal{N}_3(F)\|_{W_p^{s,\frac{1}{2}}} \lesssim \|F\|_{W_p^{s,-\frac{1}{2}}}.$$

Proof. Recall that $w(n,\tau)\lesssim \langle \tau-n^3\rangle^{0+}$. Let $\eta_k(t) = t^k\eta(t)$. First, note that $|\eta_k(t)|\leq|\eta(t)|$ since $\eta(t) = 0$ for $|t|\geq 1$. Moreover, by Hausdorff-Young and Hölder inequalities, we have $\|\langle\tau\rangle^{\frac{1}{2}+}\widehat{\eta_k}(\tau)\|_{L^p_\tau}\leq \|\eta_k\|_{H_t^{\frac{1}{2}+}}\leq\|\eta_k\|_{H^1_t}\lesssim 1+k$. Then, by Minkowski integral inequality, we have

$$\|\eta(t)\mathcal{N}_1(F)\|_{X_p^{s,\frac{1}{2}}}\leq C_\eta \sup_j \Big\|\langle n\rangle^s\int\eta(\lambda-n^3)|\widehat{F}(n,\lambda)|d\lambda\Big\|_{L^p_{|n|\sim 2^j}}\lesssim C_\eta\|F\|_{Y_p^{s,-1}},$$

where $C_\eta = \sup_n\sum_{k=1}^\infty\frac{1}{k!}\|\langle\tau-n^3\rangle^{\frac{1}{2}+}\widehat{\eta_k}(\tau-n^3)\|_{L^p_\tau}\leq\sum_{k=1}^\infty\frac{\|\langle\tau\rangle^{\frac{1}{2}+}\widehat{\eta_k}(\tau)\|_{L^p_\tau}}{k!}\lesssim\sum_{k=1}^\infty\frac{1+k}{k!}<\infty$. Similarly, we have

$$\|\eta(t)\mathcal{N}_1(F)\|_{Y_p^{s,0}}\leq C'_\eta \sup_j \Big\|\langle n\rangle^s\int\eta(\lambda-n^3)|\widehat{F}(n,\lambda)|d\lambda\Big\|_{L^p_{|n|\sim 2^j}}\lesssim C'_\eta\|F\|_{Y_p^{s,-1}},$$

where $C'_\eta = \sup_n\sum_{k=1}^\infty\frac{1}{k!}\|\widehat{\eta_k}(\tau-n^3)\|_{L^1_\tau}$. Now, note that

$$\sup_n\|\widehat{\eta_k}(\tau-n^3)\|_{L^1_\tau}\leq\sup_n\|\langle\tau-n^3\rangle^{-\frac{1}{p'}-}\|_{L^{p'}_\tau}\|\langle\tau-n^3\rangle^{\frac{1}{p'}+}\widehat{\eta_k}(\tau-n^3)\|_{L^p_\tau}\lesssim 1+k,$$

since $\frac{1}{p'}+ = 2+$. Hence, we have $C'_\eta<\infty$ as before.

For $|\tau-n^3|\gtrsim 1$, we have $|\tau-n^3|\sim\langle\tau-n^3\rangle$. Thus, we have $\widehat{\mathcal{N}_2(F)}(n,\tau)\lesssim\langle\tau-n^3\rangle^{-1}\widehat{F}(n,\tau)$. Then, by monotonicity (i.e. $\|f\|_{\widehat{W}_p^{s,\frac{1}{2}}}\leq\|g\|_{\widehat{W}_p^{s,\frac{1}{2}}}$ for $|f|\leq|g|$), we have $\|\mathcal{N}_2(F)\|_{W_p^{s,\frac{1}{2}}}\lesssim\|F\|_{W_p^{s,-\frac{1}{2}}}$.

Lastly, by Minkowski integral inequality with $w(n,\tau)\lesssim\langle\tau-n^3\rangle^{0+}$, we have

$$\begin{aligned}\|\eta(t)\mathcal{N}_3(F)\|_{X_p^{s,\frac{1}{2}}} &= \sup_j\Big\|\langle n\rangle^s\langle\tau-n^3\rangle^{\frac{1}{2}+}\widehat{\eta}(\tau-n^3)\int\frac{1-\eta(\lambda-n^3)}{\lambda-n^3}|\widehat{F}(n,\lambda)|d\lambda\Big\|_{L^p_{|n|\sim 2^j}L^P_\tau}\\ &\leq C_\eta\|F\|_{Y_p^{s,-1}},\end{aligned}$$

where $C_\eta = \sup_n\|\langle\tau-n^3\rangle^{\frac{1}{2}+}\widehat{\eta}(\tau-n^3)\|_{L^p_\tau} = \|\langle\tau\rangle^{\frac{1}{2}+}\widehat{\eta}(\tau)\|_{L^p_\tau}<\infty$. Similarly, we have

$$\|\eta(t)\mathcal{N}_3(F)\|_{Y_p^{s,0}}\lesssim C'_\eta\|F\|_{Y_p^{s,-1}},$$

where $C'_\eta = \sup_n\|\widehat{\eta}(\tau-n^3)\|_{L^1_\tau} = \|\widehat{\eta}\|_{L^1_\tau}<\infty$. $\square$

4.3. **Bilinear estimate.** By expressing (1) in the integral formulation, we see that u is a solution to (1) for $|t| \le T \ll 1$ if and only if u satisfies

$$\begin{aligned} u(t) : &= \Phi^t_{u_0}(u) \\ &= \eta(t)S(t)u_0 + \eta(t)\mathcal{N}_1(\eta_{2T}F(u))(t) + \mathcal{N}_2(\eta_{2T}F(u))(t) + \eta(t)\mathcal{N}_3(\eta_{2T}F(u))(t), \end{aligned}$$

where $F(u) = -uu_x$ and $\eta_{2T}(t) = \eta(t/2T)$, i.e. $\eta_{2T}(t) \equiv 1$ for $|t| \le T$. In this subsection, we prove the crucial bilinear estimate so that $\Phi^t_{u_0}(\cdot)$ defined above is a contraction on a ball in $W^s_p(\mathbb{T} \times [-T,T]) \subset C([-T,T], \widehat{b}^s_{p,\infty}(\mathbb{T}))$ for T sufficiently small.

Proposition 4.3. *Assume that u and v have the spatial means 0 for all $t \in \mathbb{R}$. Then, there exist $s = -\frac{1}{2}+$, $p = 2+$ with $sp < -1$, and $\theta > 0$ such that*

$$\|\eta_{2T}\partial_x(uv)\|_{W^{s,-\frac{1}{2}}_p} \lesssim T^\theta \|u\|_{W^{s,\frac{1}{2}}_p} \|v\|_{W^{s,\frac{1}{2}}_p}. \tag{26}$$

Before proving Proposition 4.3, we present some lemmata.

Lemma 4.4 (Ginibre-Tsutsumi-Velo [11], Lemma 4.2)**.** *Let $0 \le \alpha \le \beta$ and $\alpha + \beta > \frac{1}{2}$. Then, we have*

$$\int \langle \tau \rangle^{-2\alpha} \langle \tau - a \rangle^{-2\beta} d\tau \lesssim \langle a \rangle^{-\gamma},$$

where $\gamma = 2\alpha - [1-2\beta]_+$ with $[x]_+ = x$ if $x > 0$, $= \varepsilon > 0$ if $x = 0$, and $= 0$ if $x < 0$.

Lemma 4.5. *For $l_1 + 2l_2 > 1$ with $l_1, l_2 > 0$, there exists $c > 0$ such that for all $n \ne 0$ and $\lambda \in \mathbb{R}$, we have*

$$\sum_{n_1 \ne 0, n} \frac{1}{\langle n_1 \rangle^{l_1}} \frac{1}{\langle \lambda + n_1(n - n_1) \rangle^{l_2}} < c. \tag{27}$$

Proof. When $l_2 = 0$, (27) is clear. When $l_1 = 0$, (27) follows from Lemma 5.3 in [16]. Thus, we assume $l_1, l_2 > 0$ in the following. Since $l_1 + 2l_2 > 1$, there exists $\varepsilon > 0$ such that $l_1 + 2l_2 - 3\varepsilon \ge 1$.

If $P_{n,\lambda}(n_1) := \lambda + n_1(n - n_1)$ has two real roots, i.e. $P_{n,\lambda}(n_1) = -(n_1 - r_1)(n_1 - r_2)$, then there are at most 6 values of n_1 such that $|n_1 - r_j| \le 1$. For the remaining values of n_1, we have $\langle P_{n,\lambda}(n_1) \rangle > \frac{1}{4} \prod_{j=1}^2 \langle n_1 - r_j \rangle$. Then, (27) follows from Hölder inequality with $p = (l_1 - \varepsilon)^{-1}$ and $q = (l_2 - \varepsilon)^{-1}$, we have

$$\text{LHS of (27)} \lesssim \Big(\sum_{n_1} \langle n_1 \rangle^{-pl_1} \Big)^{\frac{1}{p}} \prod_{j=1}^2 \Big(\sum_{n_1} \langle n_1 - r_j \rangle^{-ql_2} \Big)^{\frac{1}{q}} < c < \infty,$$

since $pl_1 > 1$ and $ql_2 > 1$.

If $P_{n,\lambda}(n_1)$ has only one or no real root, then we have $|P_{n,\lambda}(n_1)| \ge (n_1 - \frac{1}{2}n)^2$ for all $n_1 \in \mathbb{Z}$. Then, by Hölder inequality with $p = (l_1 - \varepsilon)^{-1}$ and $q = (2l_2 - 2\varepsilon)^{-1}$, we have

$$\text{LHS of (27)} \le \Big(\sum_{n_1} \langle n_1 \rangle^{-pl_1} \Big)^{\frac{1}{p}} \Big(\sum_{n_1} \langle (n_1 - \tfrac{1}{2}n)^2 \rangle^{-ql_2} \Big)^{\frac{1}{q}} < c < \infty,$$

since $pl_1 > 1$ and $2ql_2 = \frac{l_2}{l_2 - \varepsilon} > 1$.

$\square$

Lastly, recall the following lemma from [9, (7.50) and Lemma 7.4].

Lemma 4.6. *Let*

$$\Omega(n) = \{\eta \in \mathbb{R} : \eta = -3nn_1n_2 + o(\langle nn_1n_2\rangle^{\frac{1}{100}}) \text{ for some } n_1 \in \mathbb{Z} \text{ with } n = n_1 + n_2\}.$$

Then, we have

$$\int \langle \tau - n^3\rangle^{-\frac{3}{4}} \chi_{\Omega(n)}(\tau - n^3) d\tau \lesssim 1. \tag{28}$$

Note that (28) is stated with $\langle \tau - n^3\rangle^{-1}$ in [9]. However, by examining the proof of Lemma 7.4 in [9], one immediately sees that (28) is valid with $\langle \tau - n^3\rangle^{-\alpha}$ for any $\alpha > \frac{2}{3} + \frac{1}{100}$.

Proof of Proposition 4.3. In the proof, we use (n, τ), (n_1, τ_1), and (n_2, τ_2) to denote the Fourier variables for uv, u, and v, respectively. i.e. we have $n = n_1 + n_2$ and $\tau = \tau_1 + \tau_2$ Moreover, by the mean 0 assumption on u and v and by the fact that we have $\partial_x(uv)$ on the left hand side of (26), we assume $n, n_1, n_2 \neq 0$ in the following.

First, we prove

$$\|\partial_x(uv)\|_{W_p^{s,-\frac{1}{2}}} \lesssim \|u\|_{W_p^{s,\frac{1}{2}}} \|v\|_{W_p^{s,\frac{1}{2}}}. \tag{29}$$

i.e. we first prove (26) with no gain of T^θ. Then, it suffices to show

$$\|B(f,g)(n,\tau)\|_{\widehat{W}_p^{0,-\frac{1}{2}}} \lesssim \|f\|_{b^0_{p,\infty}L^p_\tau} \|g\|_{b^0_{p,\infty}L^p_\tau}. \tag{30}$$

where $B(\cdot,\cdot)$ is defined by

$$B(f,g)(n,\tau) = \frac{1}{2\pi} \sum_{n_1+n_2=n} \frac{|n|\langle n\rangle^s}{\langle n_1\rangle^s \langle n_2\rangle^s} \int_{\tau_1+\tau_2=\tau} \frac{f(n_1,\tau_1) g(n_2,\tau_2) d\tau_1}{\prod_{j=1}^2 w(n_j,\tau_j) \langle \tau_j - n_j^3\rangle^{\frac{1}{2}}}.$$

Let $\text{MAX} := \max(\langle \tau - n^3\rangle, \langle \tau_1 - n_1^3\rangle, \langle \tau_2 - n_2^3\rangle)$. Then, by (20), we have $\text{MAX} \gtrsim \langle nn_1n_2\rangle$.

• **PART 1:** First, we consider the $\widehat{X}_p^{0,-\frac{1}{2}}$ part of the $\widehat{W}_p^{0,-\frac{1}{2}}$ norm on the left hand side of (30).

• **Case (1):** $\text{MAX} = \langle \tau - n^3\rangle$. Without loss of generality, assume $|n_1| \geq |n_2|$. For fixed $n \neq 0$ and τ, let $\lambda = \frac{\tau - n^3}{3n}$ and define

$$\begin{aligned} B_{n,\tau} = \{n_1 \in \mathbb{Z} : |n_1 - r_j| \geq 1, j = 1, 2 \\ & r_j \text{ is a real root of } P_{n,\lambda}(n_1) := \lambda + n_1(n - n_1) \\ & \text{or } r_j = \frac{1}{2}n \text{ if no real root}\}. \end{aligned}$$

On $B_{n,\tau}$, we have

$$\langle \tau - n^3 + 3nn_1n_2\rangle \gtrsim \langle n\rangle \langle \lambda + n_1(n - n_1)\rangle. \tag{31}$$

∘ Subcase (1.a): On $B^c_{n,\tau}$. For $s > -\frac{1}{2}$, we have

$$\frac{|n|\langle n\rangle^s}{\langle n_1\rangle^s \langle n_2\rangle^s} \frac{1}{\text{MAX}^{\frac{1}{2}}} \lesssim \frac{1}{\langle n_2\rangle^{\frac{1}{2}+s}}. \tag{32}$$

By Lemma 4.4, we have

$$\|\langle \tau_1 - n_1^3\rangle^{-\frac{1}{2}} \langle \tau_2 - n_2^3\rangle^{-\frac{1}{2}}\|_{L^{p'}_{\tau_1}} \lesssim \langle \tau - n^3 + 3nn_1n_2\rangle^{-1+\frac{1}{p'}}.$$

Note that for fixed n and τ there are at most four values of $n_1 \in B^c_{n,\tau}$. i.e. the summation over n_1 can be replace by the $L^p_{n_1}$ norm. Then, by Hölder inequality, we have

$$\begin{aligned}\text{LHS of (30)} &\lesssim \sup_j \Big\| \sum_{n=n_1+n_2} \frac{w(n,\tau)}{\langle n_2\rangle^{\frac12+s}} \int_{\tau=\tau_1+\tau_2} \frac{f(n_1,\tau_1)g(n_2,\tau_2)d\tau_1}{\langle \tau_1-n_1^3\rangle^{\frac12}\langle \tau_2-n_2^3\rangle^{\frac12}} \Big\|_{L^p_{|n|\sim 2^j}L^p_\tau}\\ &\lesssim \sup_j \Big\| \frac{w(n,\tau)}{\langle n_2\rangle^{\frac12+s}} \|f(n_1,\cdot)\|_{L^p_\tau}\|g(n_2,\cdot)\|_{L^p_\tau}\Big\|_{L^p_{|n|\sim 2^j}L^p_{n_1}}.\end{aligned}$$

Note that $w(n,\tau) \lesssim \langle n_2\rangle^\delta$ since $|n_1| \geq |n_2|$. If $|n_1| \gg |n_2|$ and $|n| \sim 2^j$, then we have $|n_1| \sim 2^k$ where $|k-j| \leq 5$.

$$\begin{aligned}\text{LHS of (30)} &\lesssim \sup_j \bigg(\sum_{|k-j|\leq 5}\sum_{|n_1|\sim 2^k}\sum_{l=0}^\infty \sum_{|n_2|\sim 2^l} \langle n_2\rangle^{(-\frac12-s+\delta)p} \|f(n_1,\cdot)\|^p_{L^p_\tau}\|g(n_2,\cdot)\|^p_{L^p_\tau}\bigg)^{\frac1p}\\ &\lesssim \sum_{l=0}^\infty 2^{(-\frac12-s+\delta)p\,l} \sup_k \|f\|_{L^p_{|n|\sim 2^k}L^p_\tau} \sup_l \|g\|_{L^p_{|n|\sim 2^l}L^p_\tau} \lesssim \|f\|_{b^0_{p,\infty}L^p_\tau}\|g\|_{b^0_{p,\infty}L^p_\tau},\end{aligned}$$

by taking $\delta > 0$ sufficiently small such that $-\frac12 - s + \delta < 0$. Similarly, if $|n_1| \sim |n_2|$ and $|n_2| \sim 2^l$, then we have $|n_1| \sim 2^k$ where $|k-l| \leq 5$.

$$\begin{aligned}\text{LHS of (30)} &\lesssim \bigg(\sum_{l=0}^\infty \sum_{|k-l|\leq 5}\sum_{|n_1|\sim 2^k}\sum_{|n_2|\sim 2^l} \langle n_2\rangle^{(-\frac12-s+\delta)p} \|f(n_1,\cdot)\|^p_{L^p_\tau}\|g(n_2,\cdot)\|^p_{L^p_\tau}\bigg)^{\frac1p}\\ &\lesssim \sum_{l=0}^\infty 2^{(-\frac12-s+\delta)p\,l} \sup_k \|f\|_{L^p_{|n|\sim 2^k}L^p_\tau} \sup_l \|g\|_{L^p_{|n|\sim 2^l}L^p_\tau} \lesssim \|f\|_{b^0_{p,\infty}L^p_\tau}\|g\|_{b^0_{p,\infty}L^p_\tau}.\end{aligned}$$

∘ Subcase (1.b): On $B_{n,\tau}$. In this case, we have (31). Also, recall that $w(n,\tau) \lesssim \langle \tau - n^3\rangle^{0+}$. Moreover, $\langle \tau - n^3\rangle^{0+} \lesssim \max(\langle n\rangle, \langle n_2\rangle, \langle \tau - n^3 + 3nn_1n_2\rangle)^{0+}$ since either $\langle \tau - n^3\rangle \gg |nn_1n_2|$ or $\langle \tau - n^3\rangle \lesssim |nn_1n_2| \lesssim \max(\langle n\rangle^3, \langle n_2\rangle^3)$. In particular, by (31), we have

$$w(n,\tau) \lesssim (\langle n_2\rangle\langle \tau - n^3 + 3nn_1n_2\rangle)^{0+}. \tag{33}$$

By applying Hölder inequality and proceeding as before, we have

$$\text{LHS of (30)} \lesssim M \sup_j \big\|\langle n_2\rangle^{0-}\|f(n_1,\cdot)\|_{L^p_\tau}\|g(n_2,\cdot)\|_{L^p_\tau}\big\|_{L^p_{|n|\sim 2^j}L^p_{n_1}} \lesssim M\|f\|_{b^0_{p,\infty}L^p_\tau}\|g\|_{b^0_{p,\infty}L^p_\tau},$$

where

$$M = \sup_{n,\tau} \bigg\| \frac{w(n,\tau)}{\langle n_2\rangle^{\frac12+s-}\langle \tau - n^3 + 3nn_1n_2\rangle^{1-\frac{1}{p'}}}\bigg\|_{L^{p'}_{n_1}}.$$

Thus, it remains to show that $M < \infty$. By (33), (31), and Lemma 4.5, we have

$$M^{p'} \lesssim \sup_{n,\tau} \frac{1}{\langle n\rangle^{p'-1-}} \sum_{n_2} \frac{1}{\langle n_2\rangle^{(\frac12+s-)p'}\langle \lambda + n_1(n-n_1)\rangle^{p'-1-}} < \infty,$$

since $(\frac12 + s-)p' + 2(p'-1)- > 1$ for $p = 2+ < 4$ and $sp = -1-$.

Now, assume MAX $= \langle \tau_2 - n_2^3\rangle$. By symmetry, this takes care of the case when MAX $= \langle \tau_1 - n_1^3\rangle$. Note that we have $w(n,\tau) \lesssim \langle \tau - n^3\rangle^{0+}$ by a crude estimate. Thus, by duality, it suffices to show

$$\sum_{l=0}^{\infty}\bigg\| \sum_n \frac{|n|\langle n\rangle^s}{\langle n_1\rangle^s\langle n_2\rangle^s}\frac{1}{w(n_2,\tau_2)\langle\tau_2-n_2^3\rangle^{\frac{1}{2}}}\int\frac{f(n_1,\tau_1)h(n,\tau)d\tau}{\langle\tau_1-n_1^3\rangle^{\frac{1}{2}}\langle\tau-n^3\rangle^{\frac{1}{2}-}}\bigg\|_{L^{p'}_{|n_2|\sim 2^l}L^{p'}_{\tau_2}} \tag{34}$$
$$\lesssim \sup_k \|f\|_{L^p_{|n_1|\sim 2^k}L^p_{\tau_1}}\sum_{j=0}^{\infty}\|h\|_{L^{p'}_{|n|\sim 2^j}L^{p'}_\tau}.$$

For fixed $n_2\neq 0$ and τ_2, let $\lambda=\frac{\tau_2-n_2^3}{3n_2}$ and define

$$\begin{aligned}\widetilde{B}_{n_2,\tau_2}=\{n\in\mathbb{Z}:|n-r_j|\geq 1, j=1,2\\ r_j \;\text{ is a real root of } P_{n_2,\lambda}(n):=\lambda+n(n_2-n)\\ \text{or } r_j=\frac{1}{2}n_2 \;\text{ if no real root}\}.\end{aligned}$$

On $\widetilde{B}_{n_2,\tau_2}$, we have

$$\langle\tau_2-n_2^3-3nn_1n_2\rangle\gtrsim\langle n_2\rangle\langle\lambda+n(n_2-n)\rangle. \tag{35}$$

Also, note that $w(n_2,\tau_2)\lesssim\min(\langle n\rangle^\delta,\langle n_1\rangle^\delta)$ on $\widetilde{B}^c_{n_2,\tau_2}$.

• **Case (2):** $\text{MAX}=\langle\tau_2-n_2^3\rangle$ and $|n_1|\gtrsim|n_2|$. In this case, we have

$$\frac{|n|\langle n\rangle^s}{\langle n_1\rangle^s\langle n_2\rangle^s}\frac{1}{\text{MAX}^{\frac{1}{2}}}\lesssim\frac{1}{\langle n_2\rangle^{\frac{1}{2}+s}}. \tag{36}$$

∘ Subcase (2.a): On $\widetilde{B}^c_{n_2,\tau_2}$.

First, suppose $\langle\tau_2-n_2^3-3nn_1n_2\rangle\gtrsim\langle n_2\rangle^{\frac{1}{100}}$. Thus, by Lemma 4.4, we have

$$\|\langle\tau_1-n_1^3\rangle^{-\frac{1}{2}+\alpha}\langle\tau-n^3\rangle^{-\frac{1}{2}+}\|_{L^p_\tau}\lesssim\langle\tau_2-n_2^3-3nn_1n_2\rangle^{-\frac{1}{2}+\alpha+}\lesssim\langle n_2\rangle^{\frac{-1}{100}(\frac{1}{2}-\alpha)+} \tag{37}$$

for $\alpha>0$. Then, by Hölder inequality in τ followed by Young and Hölder inequalities, we have

$$\begin{aligned}\bigg\|\int\frac{f(n_1,\tau_1)h(n,\tau)d\tau}{\langle\tau_1-n_1^3\rangle^{\frac{1}{2}}\langle\tau-n^3\rangle^{\frac{1}{2}-}}\bigg\|_{L^{p'}_{\tau_2}}&\lesssim\langle n_2\rangle^{\frac{-1}{100}(\frac{1}{2}-\alpha)+}\bigg\|\frac{f(n_1,\tau_1)}{\langle\tau_1-n_1^3\rangle^{\alpha}}h(n,\tau)\bigg\|_{L^{p'}_{\tau_2,\tau}}\\ &\leq\langle n_2\rangle^{\frac{-1}{100}(\frac{1}{2}-\alpha)+}\|\langle\tau_1-n_1^3\rangle^{-\alpha}\|_{L^{\frac{p}{p-2}}_{\tau_1}}\|f(n_1,\cdot)\|_{L^p_{\tau_1}}\|h(n,\cdot)\|_{L^{p'}_\tau}\end{aligned}$$

for fixed n and n_1. By choosing $\alpha>\frac{p-2}{p}=0+$, we have $\|\langle\tau_1-n_1^3\rangle^{-\alpha}\|_{L^{\frac{p}{p-2}}_{\tau_1}}<C<\infty$, independently of n_1.

Note that if $|n|\sim 2^j$ and $|n_2|\sim 2^l$, then we have $|n_1|\sim 2^k$ where $|k-j|\leq 5$ or $|k-l|\leq 5$ since $n=n_1+n_2$ and $|n_1|\geq|n_2|$. As in Subcase (1.a), for fixed n_2 and τ_2 there are at most four values of $n\in\widetilde{B}^c_{n_2,\tau_2}$. i.e. the summation over n can be replace by the $L^{p'}_n$ norm.

By Hölder inequality in n_2 after switching the order of summations,

$$
\begin{aligned}
\text{LHS of (34)} &\lesssim \sum_{l=0}^{\infty} \Big\| \langle n_2\rangle^{-\frac{1}{2}-s-\frac{1}{100}(\frac{1}{2}-\alpha)+} \|f(n_1,\cdot)\|_{L^p_{\tau_1}} \|h(n,\cdot)\|_{L^{p'}_\tau} \Big\|_{L^{p'}_{|n_2|\sim 2^l} L^{p'}_n} \\
&\lesssim \Big(\sum_{l=0}^{\infty}(2^l)^{0-}\Big) \sup_l \Big(\sum_{j=0}^{\infty} \sum_{|n|\sim 2^j} \| \langle n_2\rangle^{-\frac{1}{2}-s-\frac{1}{100}(\frac{1}{2}-\alpha)+}\|^{p'}_{L^{\frac{p}{p-2}}_{n_2}} \\
&\qquad \times \|f(n-n_2,\cdot)\|^{p'}_{L^p_{|n_2|\sim 2^l}L^p_{\tau_1}} \|h(n,\cdot)\|^{p'}_{L^{p'}_\tau}\Big)^{\frac{1}{p'}} \\
&\lesssim \widetilde{M} \|f\|_{b^0_{p,\infty}L^p_\tau} \|h\|_{b^0_{p',1}L^{p'}_\tau},
\end{aligned}
\tag{38}
$$

where $\widetilde{M} = \|\langle n_2\rangle^{-\frac{1}{2}-s-\frac{1}{100}(\frac{1}{2}-\alpha)+}\|_{L^{\frac{p}{p-2}}_{n_2}} < \infty$, since $\big(\frac{1}{2}+s+\frac{1}{100}(\frac{1}{2}-\alpha)-\big)\frac{p}{p-2} > 1$ for $p < \frac{2-}{1-\frac{1}{100}+} \sim \frac{200-}{99}$ with $sp < -1$. Note that we did not make use of $w(n_2,\tau_2)$ in this case.

Now, suppose $\langle \tau_2 - n_2^3 - 3nn_1n_2\rangle \ll \langle n_2\rangle^{\frac{1}{100}}$. In this case, we can not expect any contribution from $\langle \tau_2 - n_2^3 - 3nn_1n_2\rangle$ in (37). However, as long as we gain a small power of $\langle n_2\rangle$ in the denominator of LHS of (34), we can proceed as before. Note that $w(n_2,\tau_2) \sim \langle n\rangle^\delta$ since $|n_1| \gtrsim |n_2|$ implies $|n| \lesssim |n_1|$. If $|n_2| \lesssim |n|^{100}$, then we have $w(n_2,\tau_2) \gtrsim \langle n_2\rangle^{\frac{\delta}{100}}$. Otherwise, we have $|n_1| \gtrsim |n_2| \gg |n|^{100}$. Then, instead of (36), we have

$$
\frac{|n|\langle n\rangle^s}{\langle n_1\rangle^s\langle n_2\rangle^s}\frac{1}{\text{MAX}^{\frac{1}{2}}} \lesssim \frac{1}{\langle n_1\rangle^{(\frac{1}{2}+s)\frac{99}{100}}\langle n_2\rangle^{\frac{1}{2}+s}} \lesssim \frac{1}{\langle n_2\rangle^{\frac{1}{2}+s+\varepsilon}}
$$

for some $\varepsilon = 0+$. Hence, we obtain a small power of $\langle n_2\rangle$ in either case.

∘ Subcase (2.b): On $\widetilde{B}_{n_2,\tau_2}$. In this case, we have (35). As in Subcase (2.a), choose small $\alpha > \frac{p-2}{p} = 0+$. By Hölder inequality with (37) and (35), we have

$$
\int \frac{f(n_1,\tau_1)h(n,\tau)d\tau}{\langle \tau_1-n_1^3\rangle^{\frac{1}{2}}\langle\tau-n^3\rangle^{\frac{1}{2}-}} \lesssim \langle n_2\rangle^{-\frac{1}{2}+\alpha+}\langle \lambda + n(n_2-n)\rangle^{-\frac{1}{2}+\alpha+}\Big\|\frac{f(n_1,\tau_1)}{\langle\tau_1-n_1^3\rangle^\alpha}h(n,\tau)\Big\|_{L^{p'}_\tau}
$$

for fixed n, n_2, and τ_2 with $\lambda = \frac{\tau_2-n_2^3}{3n_2}$. Now by (36) and Hölder inequality in n and then in τ_1, we have

$$
\begin{aligned}
\text{LHS of (34)} &\lesssim \sum_{l=0}^{\infty}(2^l)^{0-}\widetilde{M}_1 \Big\| \langle n_2\rangle^{-1+\alpha-s+} \Big\| \frac{f(n_1,\tau_1)}{\langle\tau_1-n_1^3\rangle^\alpha}h(n,\tau)\Big\|_{L^{p'}_{\tau_2,\tau}}\Big\|_{L^{p'}_{|n_2|\sim 2^l}L^{p'}_n} \\
&\lesssim \sup_l \widetilde{M}_1 \Big\|\langle n_2\rangle^{-1+\alpha-s+}\|\langle\tau_1-n_1^3\rangle^{-\alpha}\|_{L^{\frac{p}{p-2}}_{\tau_1}}\|f(n_1,\cdot)\|_{L^p_{\tau_1}}\|h(n,\cdot)\|_{L^{p'}_\tau}\Big\|_{L^{p'}_{|n_2|\sim 2^l}L^{p'}_n},
\end{aligned}
$$

where $\widetilde{M}_1 = \sup_{n_2,\tau_2}\|\langle\lambda+n(n_2-n)\rangle^{-\frac{1}{2}+\alpha+}\|_{L^p_n} < \infty$ in view of Lemma 4.5 since $2\cdot(\frac{1}{2}-\alpha-)p > 1$. We also have $\|\langle\tau_1-n_1^3\rangle^{-\alpha}\|_{L^{\frac{p}{p-2}}_{\tau_1}} < C < \infty$, independently of n_1 as before.

Note that if $|n| \sim 2^j$ and $|n_2| \sim 2^l$, then we have $|n_1| \sim 2^k$ where $|k-j| \leq 5$ or $|k-l| \leq 5$ since $n = n_1 + n_2$ and $|n_1| \gtrsim |n_2|$. Then, by Hölder inequality in n_2, we have

$$\begin{aligned}
\text{LHS of (34)} &\lesssim \widetilde{M}_2 \sup_l \bigg(\sum_{j=0}^\infty \sum_{|n|\sim 2^j} \|f(n_1, \cdot)\|^{p'}_{L^p_{|n_1|\sim 2^k} L^p_{\tau_1}} \|h(n,\cdot)\|^{p'}_{L^{p'}_\tau} \bigg)^{\frac{1}{p'}} \\
&\lesssim \widetilde{M}_2 \|f\|_{b^0_{p,\infty} L^p_\tau} \|h\|_{b^0_{p',1} L^{p'}_\tau},
\end{aligned}$$

where $\widetilde{M}_2 = \|\langle n_2\rangle^{-1+\alpha-s+}\|_{L^{\frac{p}{p-2}}_{n_2}} < \infty$ since $(1-\alpha+s-)\frac{p}{p-2} > 1$.

• **Case (3):** $\text{MAX} = \langle \tau_2 - n_2^3 \rangle$ and $|n_1| \ll |n_2|$. $\Longrightarrow |n_1| \ll |n_2| \sim |n|$.
In this case, we have

$$\frac{|n|\langle n\rangle^s}{\langle n_1\rangle^s \langle n_2\rangle^s} \frac{1}{\text{MAX}^{\frac{1}{2}}} \lesssim \frac{1}{\langle n_1\rangle^{\frac{1}{2}+s}}. \tag{39}$$

∘ Subcase (3.a): On $\widetilde{B}^c_{n_2,\tau_2}$.

If $\langle \tau_2 - n_2^3 - 3nn_1n_2\rangle \gtrsim \langle n_2\rangle^{\frac{1}{100}}$, then we have $\langle \tau_2 - n_2^3 - 3nn_1n_2\rangle \gg \langle n_1\rangle^{\frac{1}{100}}$. By repeating the computation in Subcase (2.a), we now have a small negative power of $\langle n_1\rangle = \langle n - n_2\rangle$ in (38), instead of $\langle n_2\rangle$, which is still summable in $L^{\frac{p}{p-2}}_{n_2}$ for each fixed n. Note that if $|n_2| \sim 2^l$, then we have $|n_1| \sim 2^k$ and $|n| \sim 2^j$ where $k = 0, \cdots, l$ and $|j - l| \leq 5$. Then, it suffice to see

$$\begin{aligned}
\sum_{l=0}^\infty \|\langle n_1\rangle^{0-} F(n,n_2)\|_{L^{p'}_{|n_2|\sim 2^l} L^{p'}_n} &\lesssim \sum_{j=0}^\infty \sum_{|j-l|\leq 5} \sum_{k=0}^l (2^k)^{0-} \|F(n, n-n_1)\|_{L^{p'}_{|n|\sim 2^j} L^{p'}_{|n_1|\sim 2^k}} \\
&\lesssim \sum_{j=0}^\infty \sup_k \|F(n,n-n_1)\|_{L^{p'}_{|n|\sim 2^j} L^{p'}_{|n_1|\sim 2^k}}.
\end{aligned} \tag{40}$$

Now, suppose $\langle \tau_2 - n_2^3 - 3nn_1n_2\rangle \ll \langle n_2\rangle^{\frac{1}{100}}$. Then, we have $w(n_2,\tau_2) \sim \langle n_1\rangle^\delta$ since $|n_1| \ll |n|$. This extra gain of $\langle n_1\rangle^\delta$ in the denominator of (34) lets us proceed as before.

∘ Subcase (3.b): On $\widetilde{B}_{n_2,\tau_2}$. In this case, we have (35) and we can basically proceed as in Subcase (2.b) with (39) in place of (36). Using (40), the modification is straightforward and we omit the details.

• **PART 2:** Next, we consider the $\widehat{Y}^{0,-1}_p$ part of the $\widehat{W}^{0,-\frac{1}{2}}_p$ norm on the left hand side of (30). Define the bilinear operator $\mathcal{B}_{\theta,b}(\cdot,\cdot)$ by

$$\mathcal{B}_{\theta,b}(f,g)(n,\tau) = \frac{1}{2\pi} \sum_{n=n_1+n_2} \frac{1}{\langle \tau - n^3\rangle^\theta} \int_{\tau=\tau_1+\tau_2} \frac{|n|\langle n\rangle^s}{\langle n_1\rangle^s\langle n_2\rangle^s} \frac{f(n_1,\tau_1)g(n_2,\tau_2)d\tau_1}{\prod_{j=1}^2 w(n_j,\tau_j)\langle \tau_j - n_j^3\rangle^b}.$$

If $\text{MAX} = \langle \tau_1 - n_1^3\rangle$ or $\langle \tau_2 - n_2^3\rangle$, then by Hölder inequality, we have

$$\begin{aligned}
\text{LHS of (30)} &= \sup_j \|\mathcal{B}_{-1,\frac{1}{2}}(f,g)(n,\tau)\|_{L^p_{|n|\sim 2^j} L^1_\tau} \\
&\leq \sup_j \Big\| \|\langle \tau - n^3\rangle^{-\frac{1}{2}-\varepsilon}\|_{L^{p'}_\tau} \|\mathcal{B}_{-\frac{1}{2}+\varepsilon,\frac{1}{2}}(f,g)(n,\tau)\|_{L^p_\tau} \Big\|_{L^p_{|n|\sim 2^j}} \\
&\lesssim \sup_j \|\mathcal{B}_{-\frac{1}{2}+\varepsilon,\frac{1}{2}}(f,g)(n,\tau)\|_{L^p_{|n|\sim 2^j} L^p_\tau},
\end{aligned}$$

where we choose $\varepsilon > 0$ such that $(\frac{1}{2}+\varepsilon)p' > 1$. For $p = 2+$, we can take $\varepsilon = 0+$. Then, the proof reduces to Cases (2) and (3), where $\langle \tau - n^3\rangle^{\frac{1}{2}}$ is replaced by $\langle \tau - n^3\rangle^{\frac{1}{2}-\varepsilon}$. Note that this does not affect the argument in Cases (2) and (3).

Now, assume MAX $= \langle \tau - n^3\rangle$. If $\max(\langle \tau_1 - n_1^3\rangle, \langle \tau_2 - n_2^3\rangle) \gtrsim \langle \tau - n^3\rangle^{\frac{1}{100}}$, then by Hölder inequality, we have

$$\begin{aligned}\text{LHS of (30)} &\le \sup_j \big\| \|\langle \tau - n^3\rangle^{-\frac{1}{2}-\varepsilon}\|_{L^{p'}_\tau} \|\mathcal{B}_{-\frac{1}{2},\frac{1}{2}-100\varepsilon}(f,g)(n,\tau)\|_{L^p_\tau}\big\|_{L^p_{|n|\sim 2^j}}\\ &\lesssim \sup_j \|\mathcal{B}_{-\frac{1}{2},\frac{1}{2}-100\varepsilon}(f,g)(n,\tau)\|_{L^p_{|n|\sim 2^j}L^p_\tau}.\end{aligned}$$

Then, the proof reduces to Case (1) with $\langle \tau_j - n_j^3\rangle^{\frac{1}{2}}$ replaced by $\langle \tau_j - n_j^3\rangle^{\frac{1}{2}-100\varepsilon}$, which does not affect the argument.

Lastly, if $\max(\langle \tau_1 - n_1^3\rangle, \langle \tau_2 - n_2^3\rangle) \ll \langle \tau - n^3\rangle^{\frac{1}{100}}$, then by Hölder inequality, we have

$$\begin{aligned}\text{LHS of (30)} &\le \sup_j \big\| \|\langle \tau - n^3\rangle^{-\frac{1}{2}}\chi_{\Omega(n)}(\tau - n^3)\|_{L^{p'}_\tau} \|\mathcal{B}_{-\frac{1}{2},\frac{1}{2}}(f,g)(n,\tau)\|_{L^p_\tau}\big\|_{L^p_{|n|\sim 2^j}}\\ &\lesssim \sup_j \|\mathcal{B}_{-\frac{1}{2},\frac{1}{2}}(f,g)(n,\tau)\|_{L^p_{|n|\sim 2^j}L^p_\tau},\end{aligned}$$

where the second inequality follows from Lemma 4.6 since $-\frac{1}{2}p' = -1+ < -\frac{3}{4}$. Once again, the proof reduces to Case (1).

• **PART 3:** In this last part, we discuss how to gain a small power of T in (26) by assuming that u or v are supported locally in time. In Part 1 and 2, we indeed showed

$$\|\partial_x(uv)\|_{W_p^{s,-\frac{1}{2}}} \lesssim \|\widehat{u}\|_{\widehat{X}_p^{s,b}}\|w\widehat{v}\|_{\widehat{X}_p^{s,\frac{1}{2}}} + \|w\widehat{u}\|_{\widehat{X}_p^{s,\frac{1}{2}}}\|\widehat{v}\|_{\widehat{X}_p^{s,b}} \tag{41}$$

for some $b \in (0,\frac{1}{2})$ since we needed the full power of $\frac{1}{2}$ from only one of $\langle \tau - n^3\rangle$, $\langle \tau_1 - n_1^3\rangle$, or $\langle \tau_2 - n_2^3\rangle$, i.e. from the maximum one, and the weight $w(n_j,\tau_j)$ was needed only when MAX $= \langle \tau_j - n_j^3\rangle$. Thus, (26) follows once we prove

$$\|\eta_{2T}u\|_{X_p^{s,b}} \lesssim T^\theta \|u\|_{X_p^{s,\frac{1}{2}}} \tag{42}$$

for some $\theta > 0$. By interpolation, we have

$$\|u\|_{X_p^{s,b}} \lesssim \|u\|^\alpha_{X_p^{s,0}}\|u\|^{1-\alpha}_{X_p^{s,\frac{1}{2}}}, \tag{43}$$

where $\alpha = 1-2b \in (0,1)$. Recall $\widehat{\eta_{2T}}(\tau) = 2T\widehat{\eta}(2T\tau)$. Hence, we have $\|\widehat{\eta_{2T}}\|_{L^q_\tau} \sim T^{\frac{q-1}{q}}\|\widehat{\eta}\|_{L^q_\tau} \sim T^{\frac{q-1}{q}}$. i.e. we can gain a positive power of T as long as $q > 1$. For fixed n, by Young and Hölder inequalities, we have

$$\|\widehat{\eta_{2T}} * \widehat{u}(n,\cdot)\|_{L^p_\tau} \le \|\widehat{\eta_{2T}}\|_{L^{p'}_\tau}\|\widehat{u}(n,\cdot)\|_{L^{\frac{p}{2}}_\tau} \lesssim T^{\frac{p'-1}{p'}}\|\langle \tau - n^3\rangle^{-\frac{1}{2}}\|_{L^p_\tau}\|\langle \tau - n^3\rangle^{\frac{1}{2}}\widehat{u}(n,\cdot)\|_{L^p_\tau}$$

Hence, for $p > 2$, we have

$$\|u\|_{X_p^{s,0}} \lesssim T^{\frac{1}{p}}\|u\|_{X_p^{s,\frac{1}{2}}}. \tag{44}$$

Then, (42) follows from (43) and (44). This completes the proof. □

Remark 4.7. A simple modification of the proof of Proposition 4.3 can be used to establish the local well-posedness of (1) in $\mathcal{F}L^{s,p} = \widehat{b}^s_{p,p}$ for some $p = 2+$, $s = -\frac{1}{2}+$ with $sp < -1$ as well. Such local solutions can be extended globally a.s. on the statistical ensemble from the discussion in Section 3. The modification is straightforward and we omit the details.

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References

[1] I. Bejenaru, T. Tao, *Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation,* J. Funct. Anal. 233 (2006), 228–259.

[2] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations II*, GAFA., 3 (1993), 209–262.

[3] J. Bourgain, *Periodic Korteweg-de Vries equation with measures as initial data,* Sel. Math., New Ser. 3 (1997) 115–159.

[4] J. Bourgain, *Periodic nonlinear Schrödinger equation and invariant measures,* Comm. Math. Phys. 166 (1994), 1–26.

[5] J. Bourgain, *On the Cauchy and invariant measure problem for the periodic Zakharov system,* Duke Math. J. 76 (1994), 175–202.

[6] N. Burq, N. Tzvetkov, *Invariant measure for a three dimensional nonlinear wave equation,* Int. Math. Res. Not. (2007), no. 22, Art. ID rnm108, 26pp.

[7] S. Cambronero, H. P. McKean, *The ground state eigenvalue of Hill's equation with white noise potential,* Comm. Pure Appl. Math. 52 (1999) no10, 1277–1294.

[8] M. Christ, J. Colliander, T. Tao, *Asymptotics, frequency modulation, and low-regularity illposedness of canonical defocusing equations,* Amer. J. Math. 125 (2003), no. 6, 1235–1293.

[9] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Sharp Global Well-Posedness for KdV and Modified KdV on* $\mathbb{R}$ *and* $\mathbb{T}$, J. Amer. Math. Soc. 16 (2003), no. 3, 705–749.

[10] M.X. Fernique, *Intégrabilité des Vecteurs Gaussiens,* Academie des Sciences, Paris, Comptes Rendus, 270, Séries A (1970), 1698–1699.

[11] J. Ginibre, Y. Tsutsumi, G. Velo, *On the Cauchy Problem for the Zakharov System,* J. Funct. Anal., 151 (1997), 384–436.

[12] L. Gross, *Abstract Wiener spaces,* Proc. 5th Berkeley Sym. Math. Stat. Prob. 2 (1965), 31–42.

[13] T. Kappeler and P. Topalov, *Global wellposedness of KdV in* $H^{-1}(\mathbb{T}, \mathbb{R})$, Duke Math. J. 135 (2006), no. 2, 327–360.

[14] C. Kenig, G. Ponce, and L. Vega, *A bilinear estimate with applications to the KdV equation,* J. Amer. Math. Soc. 9 (1996), no. 2 573–603.

[15] C. Kenig, G. Ponce, L. Vega, *On the ill-posedness of some canonical dispersive equations,* Duke Math. J. 106 (2001), no.3, 617–633.

[16] C. Kenig, G. Ponce, and L. Vega, *Quadratic forms for the 1-D semilinear Schrödinger equation,* Trans. Amer. Math. Soc., 348 (1996), 3323–3353.

[17] N. Kishimoto, *Local well-posedness for the Cauchy problem of the quadratic Schrödinger equation with nonlinearity* $\overline{u}^2$, to appear in Comm. Pure Appl. Anal.

[18] H. Kuo, *Gaussian Measures in Banach Spaces,* Lec. Notes in Math. 463, Springer-Verlag, New York, 1975.

[19] T. Oh, *Invariant Gibbs measures and a.s. global well-posedness for coupled KdV systems,* to appear in Diff. Integ. Equations.

[20] T. Oh, *Invariance of the Gibbs Measure for the Schrödinger-Benjamin-Ono System,* preprint.

[21] J. Quastel, B. Valkó, *KdV preserves white noise,* Comm. Math. Phys. 277 (2008), no. 3, 707–714.

[22] B. Roynette, *Mouvement brownien et espaces de Besov,* Stochastics Stochastics Rep. 43 (1993), 221–260.

[23] N. Tzvetkov, *Invariant measures for the nonlinear Schrödinger equation on the disc,* Dyn. Partial Differ. Equ. 3 (2006), no. 2, 111–160.

[24] N. Tzvetkov, *Invariant measures for the defocusing Nonlinear Schrödinger equation (Mesures invariantes pour l'équation de Schrödinger non linéaire),* Annales de l'Institut Fourier, 58 (2008), 2543–2604.

[25] P. Zhidkov, *Korteweg-de Vries and Nonlinear Schrödinger Equations: Qualitative Theory,* Lec. Notes in Math. 1756, Springer-Verlag, 2001.

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